Method For Boson Spin Determination

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The reaction $K(\pi)+P\rightarrow Y+B$ is considered, in which Y is a spin-$\frac{1}{2}$ hyperon and B is a spin-ℓ boson which decays into two spin-zero bosons. Taking into account the consequences of parity conservation in the production process, the structure of the joint decay distribution of Y and B is defined for arbitrary spin ℓ. It is shown that, at given production angle and total center-of-mass energy, there are $2\ell(2\ell-1)$ constraints on the complete set of moments of the joint decay distribution. These constraints may be utilized to test for the compatibility of a set of data with each hypothesized value of ℓ. The structure of the joint decay distribution may also be used to define the forbidden moments, whose absence may be experimentally verified in order to rule out distortion of the distribution due to interference effects, final-state interaction, or experimental biases. The absence of the forbidden moments is also important in cases where the spin of B is known and the data are used to determine the production parameters.

IN reactions of the type,

$$K(\pi)+P\rightarrow Y+B \searrow B_0+B_0', \tag{1}$$

where Y is a spin-$\frac{1}{2}$ hyperon and B is a spin-ℓ boson which decays into two spin-zero bosons B_0, Eberhard and Good[1] demonstrated that the correlations between the decay directions of Y and B contain information which has bearing on the determination of ℓ. They derived for each nonzero spin ℓ, a specific inequality that quadratic forms of the data must satisfy at each production angle (Θ) and total center-of-mass energy (E^*). The consequences of parity conservation in the production process were not used, and the polarization of Y is only examined along one (arbitrary) direction.

In this paper we show that if parity conservation in the production process is taken into account, and if all three components of the hyperon spin are used, at given Θ and E^*, there are *$2\ell(2\ell-1)$ constraints* on the moments of the decay distributions. These constraints may be utilized to test for the compatibility of a set of data with each hypothesized ℓ. The calculation based only on invariance arguments, for general spin ℓ,[2] is presented and the structure of the joint decay distribution of Y and B is defined.

A description of the final state of reaction (1) which is compatible with angular momentum and parity conservation in the production process is facilitated if use is made of Bohr's theorem on reflection invariance.[3] With the quantization axis chosen to be along the normal to the production plane ($\hat{n}\sim\hat{K}\times\hat{B}$), the following relation holds between the intrinsic parities and the magnetic quantum numbers (m) of all the particles participating in the reaction:

$$\exp[i\pi\sum m_i]=\mathcal{P}\exp[i\pi\sum m_f], \tag{2}$$

where the sums $\sum m_i$ and $\sum m_f$ are over the particles in the initial and final states, respectively, and $\mathcal{P}$ is the product of the intrinsic parities of all the particles.

Equation (2) allows the final state of reaction (1) to be described by the following two incoherent wave functions resulting from target proton spin up (down) and down (up), respectively, according as $\mathcal{P}$ is odd (even)[4]:

$$\begin{aligned}\psi_1&=\sum_{\text{odd } m} a_m Y_\ell^m(\hat{k})\alpha+\sum_{\text{even } m} a_m Y_\ell^m(\hat{k})\beta,\\ \psi_2&=\sum_{\text{odd } m} b_m Y_\ell^m(\hat{k})\beta+\sum_{\text{even } m} b_m Y_\ell^m(\hat{k})\alpha,\end{aligned} \tag{3}$$

where the a_m and b_m are functions of Θ and E^*, α and β are Pauli spinors describing the Y spin state, and the argument $\hat{k}$ of the spherical harmonics is the unit decay vector of B expressed in the rest system of B. It is shown below that only the direction and not the sense of $\hat{k}$ is important here. Although the choice of azimuthal angle is arbitrary in the above discussion, for definiteness the x axis is taken to be along the beam direction so that the coordinate system is $(\hat{x},\hat{y},\hat{z})\equiv(\hat{K},\hat{n}\times\hat{K},\hat{n})$ with all vectors defined in the total center-of-mass system.

Assuming an unpolarized target, calculation of the distribution function of $\hat{k}$ averaged over Y spin $I(\hat{k})=\langle\psi_1|\psi_1\rangle+\langle\psi_2|\psi_2\rangle$ and of the Y polarization distribution $\mathbf{IP}(\hat{k})=\langle\psi_1|\boldsymbol{\sigma}|\psi_1\rangle+\langle\psi_2|\boldsymbol{\sigma}|\psi_2\rangle$ yield the following forms,

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[1] P. Eberhard and M. L. Good, Phys. Rev. **120**, 1442 (1960). Restrictions on the decays of Y and B in the Adair limit of fore or aft production angle were discussed by S. B. Treiman, Phys. Rev. **128**, 1342 (1962).

[2] These constraints have previously been pointed out for the case $\ell=1$ by N. Byers and C. N. Yang (unpublished).

[3] A. Bohr, Nucl. Phys. **10**, 486 (1959).

[4] The use of $Y_L^M(\hat{k})$ for the spin wave function of B is valid because the two final-state bosons have spin zero. It is also true in the case of a 3π decay of a 1^- particle (for example, the ω) where the matrix element has the structure $\hat{\pi}_i\times\hat{\pi}_j$ that $Y_L^M(\hat{k})$ is a suitable form of the spin wavefunction ($\hat{k}\sim\hat{\pi}_i\times\hat{\pi}_j$). Extension of the method discussed in this Letter to decays $B\rightarrow 3B_0$ of higher spin objects will require the use of other suitable angular functions.

$$I(\hat{k})=\sum_{L=0}^{2\ell}\sum_{\text{even }M}\eta_{\ell L}Y_L{}^M(\hat{k})\{\sum_m(-)^m c(\ell\ell L,\,mM-m)(a_m a_{m-M}{}^*+b_m b_{m-M}{}^*)\}\,,\tag{4a}$$

$$IP_z(\hat{k})=\sum_{L=0}^{2\ell}\sum_{\text{even }M}\eta_{\ell L}Y_L{}^M(\hat{k})\{\sum_m c(\ell\ell L,\,mM-m)(-a_m a_{m-M}{}^*+b_m b_{m-M}{}^*)\}\,,\tag{4b}$$

$$IP_y(\hat{k})=i\sum_{L=0}^{2\ell}\sum_{\text{odd }M}\eta_{\ell L}Y_L{}^M(\hat{k})\{\sum_m c(\ell\ell L,\,mM-m)(-a_m a_{m-M}{}^*+b_m b_{m-M}{}^*)\}\,,\tag{4c}$$

$$IP_x(\hat{k})=-\sum_{L=0}^{2\ell}\sum_{\text{odd }M}\eta_{\ell L}Y_L{}^M(\hat{k})\{\sum_m(-)^m c(\ell\ell L,\,mM-m)(a_m a_{m-M}{}^*+b_m b_{m-M}{}^*)\}\,,\tag{4d}$$

where

$$\eta_{\ell L}=\frac{2\ell+1}{[4\pi(2L+1)]^{1/2}}c(\ell\ell L;\,00)\,,$$

and where the notation for the Clebsch-Gordan coefficients is $c(\ell_1\ell_2L;\,m_1m_2)$. $\eta_{\ell L}=0$ if L is odd and therefore only even L $Y_L{}^M$ occur in these distributions. This is a consequence of angular-momentum conservation in the decay of B. Thus the sign of $\hat{k}$ plays no role in the analysis. The consequence of parity conservation in the production process is that only *even* M $Y_L{}^M$ occur in $I(\hat{k})$ and $IP_z(\hat{k})$ and only *odd* M $Y_L{}^M$ occur in $IP_y(\hat{k})$ and $IP_x(\hat{k})$.

I and $\mathbf{IP}$ determine the joint decay distribution function of Y and B, which may be written as,

$$\mathfrak{D}(\hat{k},\hat{p})d\Omega_k dxdydz=\tfrac{1}{8}(I(\hat{k})+\alpha_Y\mathbf{IP}(\hat{k})\cdot\hat{p})d\Omega_k dxdydz\,,\tag{5}$$

where $\hat{p}$ is the Y decay-baryon unit vector expressed in the Y rest frame, x, y, z are its direction cosines (with respect to $\hat{x}$, $\hat{y}$, $\hat{z}$), $d\Omega_k$ is the solid angle differential of $\hat{k}$, and α_Y is the asymmetry parameter in Y decay.

The moments of the experimental data which may be evaluated are $\langle Y_L{}^M(\hat{k})\rangle$, $\langle xY_L{}^M(\hat{k})\rangle$, $\langle yY_L{}^M(\hat{k})\rangle$, $\langle zY_L{}^M(\hat{k})\rangle$. For example, the last of these evaluated with Eq. (5) gives

$$\langle zY_L{}^M(\hat{k})\rangle=\int\!\!\int\mathfrak{D}(\hat{k},\hat{p})zY_L{}^M(\hat{k})d\Omega_k dxdydz=\frac{\alpha_Y}{3}\int IP_z(\hat{k})Y_L{}^M(\hat{k})d\Omega_k\,.\tag{6}$$

Equations (4a)–(4d) thus yield the following relationships[5] between the experimental moments and the production parameters of Eq. (3), where as in Eqs. (4), $0\leqslant L\leqslant 2\ell$;

even M only;

$$\langle Y_L{}^M\rangle=\eta_{\ell L}\sum_m(-)^m c(\ell\ell L,\,mM-m)(A_{mM}+B_{mM})\,,$$

$$\langle zY_L{}^M\rangle=-\left(\frac{\alpha_Y}{3}\right)\eta_{\ell L}\sum_m c(\ell\ell L;\,mM-m)(A_{mM}-B_{mM});\tag{7}$$

odd M only;

$$\langle yY_L{}^M\rangle=i\left(\frac{\alpha_Y}{3}\right)\eta_{\ell L}\sum_m c(\ell\ell L,\,mM-m)(A_{mM}-B_{mM})\,,$$

$$\langle xY_L{}^M\rangle=-\left(\frac{\alpha_Y}{3}\right)\eta_{\ell L}\sum_m(-)^m c(\ell\ell L,\,mM-m)(A_{mM}+B_{mM})\,,$$

where

$$A_{mM}=a_m{}^*a_{m-M}\,,$$
$$B_{mM}=b_m{}^*b_{m-M}\,.$$

The normalization condition $\langle Y_0{}^0\rangle=(4\pi)^{-1/2}$ requires that $\sum_m\{|a_m|^2+|b_m|^2\}=1$. An error matrix for the moment measurements in a given experiment may be calculated in the usual way using the relation $U_{XY}=(1/N(N-1))\sum_{i=1}^N(\bar{X}-X_i)(\bar{Y}-Y_i)$ which gives the covariance for moments X and Y, $\bar{X}$ and $\bar{Y}$ being the experimental averages and the subscript i referring to the ith event.

It is seen that for given spin ℓ there are $M_\ell=2(\ell+1)\times(2\ell+1)$ measurable $Y_L{}^M$ moments (counting the nor-

[5] Simpler functions of the production parameters can be projected out of Eqs. (7) by multiplying by the Clebsch-Gordan coefficient $C(\ell\ell L;\,mM-m)=\frac{1}{2}[c(\ell\ell L;\,mM-m)+c(\ell\ell L;\,M-mm)]$ and summing over L.

malization condition). Thus, for each Θ and E^*, there are M_ℓ different functions of the $N_\ell=2(4\ell+1)$ parameters of Eqs. (3). In other words, the a_m and b_m are overdetermined by $C_\ell=M_\ell-N_\ell=2\ell(2\ell-1)$ relations. For example, $C_1=12-10=2$,[2] and $C_2=30-18=12$.

For a given set of experimental data, complexity arguments may be used to find a lower limit for the spin $\ell \geqslant L_{\max}/2$. The constraints discussed in this letter may be utilized to calculate a C_ℓ degree-of-freedom χ^2 for the compatibility of the data with each hypothesized spin ℓ. For each point in the N_ℓ-dimensional space of production parameters, moments may be calculated using Eqs. (7) and a χ^2 evaluated that the experimental moments are given by these hypothesized moments. The results of a χ^2-minimization search in the N_ℓ space yields a probability that the data are compatible with the hypothesized ℓ. This procedure is then followed for all values of ℓ under consideration. A spin identification can then be made if there is only one ℓ which yields a reasonable probability.

Inasmuch as these considerations hold only for fixed Θ and E^*, it is necessary to examine the question of how one would treat data in practice. The extent to which the data may be averaged over Θ and E^* depends on the production dynamics in Eq. (1), an apriori unknown factor. However, the question may be settled empirically by noting that if averages are made over excessively large ranges of Θ and E^*, then even for the correct ℓ, $\chi_{\min}^2$ should have a very small probability, corresponding to the fact that because the moments depend quadratically on the production parameters, data that have been averaged over large ranges of Θ and E^* will no longer satisfy the above constraint equations. One should find experimentally that as the range of Θ and E^*, over which the data are averaged, is made progressively smaller, the $\chi_{\min}^2$ for (at least) the correct ℓ hypothesis should also decrease to the region where it represents a reasonable probability.

With the increasing frequency of bubble-chamber experiments involving many hundreds and even thousands of events of particular reactions under study, it now seem possible to perform analyses of data in small (i.e., $\Delta\cos\Theta \lesssim 0.1$) bins of Θ. In the Ξ^* spin-parity determination experiment of Schlein *et al.*[6] in which polarization moments of the data were evaluated, it was shown with the use of Monte Carlo techniques that χ^2 calculated with 80 events essentially followed the theoretical χ^2 distributions. χ^2 results of independent analyses of $n\Theta$ and E^* bins may be combined in the following way. If χ_{ij}^2 with C degrees-of-freedom is the result of the jth independent test of hypothesis i, then the over-all χ_i^2 for the hypothesis is given by $\chi_i^2=\sum_{j=1}^n \chi_{ij}^2$ with nC degrees-of-freedom.

Departures of the experimental decay distributions from the distributions given by Eqs. (4) and (5) may be detected by looking for the presence of "illegal" Y_L^M moments. These departures may result either from experimental biases or from interference between reaction (1) and other channels which lead to the same final state $Y+B_0+B_0'$. Interference of this type may be expected to be particularly serious in cases where B is a very broad resonance, such that its mean decay distance is of the same order as the range of nuclear forces. In cases where the spin of B is known and the data are used to determine the production parameters, the absence of the "illegal" moments, is of course, a requisite to any subsequent moment analysis.

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[6] P. E. Schlein, D. D. Carmony, G. M. Pjerrou, W. E. Slater. D. H. Stork, and H. K. Ticho, Phys. Rev. Letters **11**, 167 (1963).